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7, 8, 9, 10-TUPLE COMPLETE PARTITIONS OF INTEGERS

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Abstract

This paper presents the concepts of 7, 8, 9, 10-tuple (septuple, octuple, nonuple, decuple) complete partitions of integers and an attempt has been done for the theorem based on the last part of septuple, octuple, nonuple and decuple complete partitions of integers.

Introduction

The partition function [1] $p(n)$ is defined as the number of ways, that the positive integer n can be written as a sum of positive integers, as in $n = a_1 + a_2 + \ldots + a_r$. The summands a_j are called the parts of the partition. Although the parts need not be distinct, two partitions are not considered as

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different, if they differ only in the order of their parts. Many Mathematicians studied the unique representations of positive integers by some sequences with given properties. For example, Zeckendorf found that every integer can be uniquely represented as a sum of inconsecutive terms of Fibonacci sequences. Mac Mahon [2] studied perfect partitions of n, which are partitions of n such that every integer m with $1 \leq m \leq n$ is uniquely represented in one and only one way. In 1960, Hoggatt [3] considered sequences such that, every positive integer can be represented as a sum of some terms of the sequences and Brown [4] studied such sequences and named it as "complete" which are defined as sequences $(s_1, s_2, \ldots)$ such that every integer can be represented as $\sum_{i=1}^{\infty} \alpha_i s_i$, where $\alpha_i \in S = \{0, 1\}$. A partition which is complete was studied in [5]. This was also generalized [6] by replacing the set $S = \{0, 1\}$ by the set $S = \{0, 1, \ldots, r\}$. A complete partition of an integer n is a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of n, with $\mu_1 = 1$, such that each integer $r, 1 \leq r \leq n$, can be represented as a sum of elements of $\mu_1, \mu_2, \ldots, \mu_k$. In other words, each r can be expressed as $\sum_{i=1}^{k} \beta_j \mu_j$, where each β_j is either 0 or 1.

Hokyu Lee generalized the perfect partition and found a relation with ordered factorizations [11]. Oystein J. Rodseth presented the study of enumeration of M-partitions, weak M-partitions and generating functions [12]. Mac Mahon (2006) initiated the study of double perfect partitions and he found a relation with ordered factorizations. Oystein J. Rodseth (2007) produced the some standard results, generating functions and completeness of minimal r-complete partitions [13]. James A. Seller made significant observations on the parity of the total number of parts in odd-part partitions. George E. Andrews investigated the partitions with distinct evens and produced companion theorems for distinct evens partitions counted by exceptional parts [14].

Motivated by the above, as an amateur number theorist, an attempt has been made in a slightly lighter mode to study on higher order partitions.

Now, the 7, 8, 9, 10-tuple (septuple, octuple, nonuple, decuple) complete partitions of integers are defined [7], [8], [9], [10].

Definition 1. For any integer $n \geq 22$, its 7-tuple (septuple) complete partition can be obtained by taking the parts of n as $\mu = (\mu_1^{m_1} \mu_2^{m_2} \ldots \mu_l^{m_l})$ of n such that each integer r, with $7 \leq r \leq n - 7$ can be represented at least seven different ways as a sum $\sum_{i=1}^{l} \gamma_i \mu_i$ with $\gamma_i \in \{0, 1, 2, \ldots, m_l\}$.

In the same way the 8, 9, 10 -tuple complete partition of integers are also defined. For all 7, 8, 9, 10-tuple complete partitions of integers n should be greater than or equal to 22.

Definition 2. For any integer $n \geq 22$, its 8-tuple (octuple) complete partition can be obtained by taking the parts of n as $\mu = (\mu_1^{m_1} \mu_2^{m_2} \ldots \mu_l^{m_l})$ of n such that each integer r, with $8 \leq r \leq n - 8$ can be represented at least eight different ways as a sum $\sum_{i=1}^{l} \gamma_i \mu_i$ with $\gamma_i \in \{0, 1, 2, \ldots, m_l\}$.

Definition 3. For any integer $n \geq 22$, its 9-tuple (nonuple) complete partition can be obtained by taking the parts of n as $\mu = (\mu_1^{m_1} \mu_2^{m_2} \ldots \mu_l^{m_l})$ of n such that each integer r, with $9 \leq r \leq n - 9$ can be represented at least nine different ways as a sum $\sum_{i=1}^{l} \gamma_i \mu_i$ with $\gamma_i \in \{0, 1, 2, \ldots, m_l\}$.

Definition 4. For any integer $n \geq 22$, its 10-tuple (decuple) complete partition can be obtained by taking the parts of n as $\mu = (\mu_1^{m_1} \mu_2^{m_2} \ldots \mu_l^{m_l})$ of n such that each integer r, with $10 \leq r \leq n - 10$ can be represented at least ten different ways as a sum $\sum_{i=1}^{l} \gamma_i \mu_i$ with $\gamma_i \in \{0, 1, 2, \ldots, m_l\}$.

Theorem 1.1. *If a partition* $\mu = (\mu_1^{m_1} \mu_2^{m_2} \ldots \mu_l^{m_l})$ *of a positive integer* $n \geq 22$ *is a* 7-*tuple* (*or* 8, 9, 10-*tuple*) *complete partition then* $\mu_{i+1} \leq \sum_{i=1}^{l} m_j \mu_j - 9$ *with* $i \geq 5$ *and* μ *should have at least two* 1*'s*, *one* 2, *one* 3, *one* 4, *one* 5 *and one* 6 (*or*) *one* 1, *two* 2*'s*, *one* 3, *one* 4, *one* 5 *and one* 6 (*or*) *one* 1, *one* 2, *two* 3'*s*, *one* 4, *one* 5 *and one* 6 (*or*) *one* 1, *one* 2, *one* 3, *two* 4'*s*, *one* 5 *and one* 6 (*or*) *one* 1, *one* 2, *one* 3, *one* 4, *one* 5 *and two* 6'*s as its parts*.

Proof. For any integer n, its 7-tuplecomplete partition can be obtained by taking the value as $n \geq 22$, and the parts of the integer n should be equivalent to $(\mu_1^{m_1}\mu_2^{m_2} \ldots \mu_l^{m_l})$. We can prove this theorem by considering the parts of the integer as $n = \mu_1^{m_1}\mu_2^{m_2}\mu_3^{m_3}\mu_4^{m_4}\mu_5^{m_5}\mu_5^{m_5}\mu_6^{m_6}$. That is, $n = 1^{m_1}2^{m_2}$ $3^{m_3}4^{m_4}5^{m_5}6^{m_6}$ with $m_1 \geq 2, m_2, m_3, m_4, m_5$ and $m_6 \geq 1$ and $\mu_6 \leq m_1 + m_2 + m_3 + m_4 + m_5$ is a 7-tuple complete partition of the integer $n = m_1\mu_1 + m_2\mu_2 + m_3\mu_3 + m_4\mu_4 + m_5\mu_5 + m_6\mu_6$. If it is a 7-tuple complete partition, then for every integer $r, 7 \leq r \leq \sum_{j=1}^{i} m_j\mu_j - 7$ can be written as seven different ways using the parts 1, 2, 3, 4, 5 and 6. Therefore, $m_2\mu_1 + m_3\mu_6$, $m_1\mu_1 + m_3\mu_5$, $m_2\mu_2 + m_5\mu_5$, $m_3\mu_3 + m_4\mu_4$, $m_1\mu_2 + m_3\mu_3$, $m_2\mu_1 + m_1\mu_3$, and $m_1\mu_1 + m_2\mu_2 + m_3\mu_3$ are the seven representations of n with μ satisfies the condition $\mu_{i+1} \leq \sum_{j=1}^{i} m_j\mu_j - 9$. Now we check the condition $\mu_{i+1} \leq \sum_{j=1}^{i} m_j\mu_j - 9$ for n. Let us assume that $n = \mu_1^{m_1}\mu_2\mu_3\mu_4\mu_5\mu_6$, $n = \mu_1 \mu_2^{m_2}\mu_3\mu_4\mu_5\mu_6$, $n = \mu_1\mu_2\mu_3^{m_3}\mu_4\mu_5\mu_6$, $n = \mu_1\mu_2\mu_3\mu_4^{m_4}\mu_5\mu_6$, $n = \mu_1\mu_2\mu_3\mu_4\mu_5^{m_5}\mu_6$ and $n = \mu_1\mu_2\mu_3\mu_4\mu_5\mu_6^{m_6}$ be a 7-tuple complete partitions of n with $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 2$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 3, \mu_4 = 4, \mu_5 = 5, \mu_6 = 6$ (1).

Case (i)**.** If we take $n = \mu_1^{m_1}\mu_2\mu_3\mu_4\mu_5\mu_6$, is a 7-tuple complete partition then it should satisfy the condition $\mu_{i+1} \leq \sum_{j=1}^{i} m_j\mu_j - 9$ (2) with $i = 5$. Here $n = 22$ by equation (1) and $6 \leq 7$. Therefore, μ satisfies the condition.

Case (ii)**.** If we take $n = \mu_1\mu_2^{m_2}\mu_3\mu_4\mu_5\mu_6$, is a 7-tuple complete partition then by equations (1) and (2) $n = 23$ and $6 \leq 8$.

Case (iii)**.** If we take $n = \mu_1\mu_2\mu_3^{m_3}\mu_4\mu_5\mu_6$, is a 7-tuple complete partition then by equations (1) and (2) $n = 24$ and $6 \le 9$.

Case (iv)**.** If we take $n = \mu_1\mu_2\mu_3\mu_4^{m_4}\mu_5\mu_6$, is a 7-tuple complete partition then by equations (1) and (2) $n = 25$ and $6 \le 10$.

Case (v)**.** If we take $n = \mu_1\mu_2\mu_3\mu_4\mu_5^{m_5}\mu_6$ is a 7-tuple complete partition then by equations (1) and (2) $n = 26$ and $6 \le 11$.

Case (vi)**.** If we take $n = \mu_1\mu_2\mu_3\mu_4\mu_5\mu_6^{m_6}$ is a 7-tuple complete partition then by equations (1) and (2) $n = 27$ and $6 \le 6$.

If the above cases are true for $n = \mu_1^{m_1}\mu_2\mu_3\mu_4\mu_5\mu_6$, $n = \mu_1\mu_2^{m_2}\mu_3\mu_4\mu_5\mu_6$, $n = \mu_1\mu_2\mu_3^{m_3}\mu_4\mu_5\mu_6$, $n = \mu_1\mu_2\mu_3\mu_4^{m_4}\mu_5\mu_6$, $n = \mu_1\mu_2\mu_3\mu_4\mu_5^{m_5}\mu_6$ and $n = \mu_1\mu_2\mu_3\mu_4\mu_5\mu_6^{m_6}$ then the condition $\mu_{i+1} \le \sum_{j=1}^{i} m_j\mu_j - 9$ is also true for $n = (\mu_1^{m_1}\mu_2^{m_2} \ldots \mu_l^{m_l})$. Hence μ satisfies the condition $\mu_{i+1} \le \sum_{j=1}^{i} m_j\mu_j - 9$.

Corollary 1.2. *Let* $\mu = (\mu_1^{m_1}\mu_2^{m_2} \ldots \mu_l^{m_l})$ *be a* 7*-tuple complete partition of a positive integer n. Then* $\mu_{i+1} \le \sum_{j=1}^{i} 7^{j-1}\mu_j$ *where* μ_{i+1} *is the last part of the* 7*-tuple complete partition of an integer.*

Proof. In a 7-tuple complete partition, $n \ge 22$ and $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ and μ_6 should be equivalent to 1, 2, 3, 4, 5 and 6 respectively.

$$\mu_{i+1} \le \mu_1 + \mu_2 + \ldots + \mu_j \le 7\mu_1 + 7\mu_2 + \ldots + 7\mu_j \le 7^{j-1}\mu_1 + 7^{j-1}\mu_2 + \ldots + 7^{j-1}\mu_j$$

$$\le \sum_{j=1}^{i} 7^{j-1}\mu_j.$$

Corollary 1.3. *Let* $\mu = (\mu_1^{m_1}\mu_2^{m_2} \ldots \mu_l^{m_l})$ *be a* 8*-tuple complete partition of a positive integer n. Then* $\mu_{i+1} \le \sum_{j=1}^{i} 8^{j-1}\mu_j$ *where* μ_{i+1} *is the last part of the* 8*-tuple complete partition of an integer.*

Proof. In a 8-tuple complete partition, $n \ge 22$ and $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ and

μ_6 should be equivalent to 1, 2, 3, 4, 5 and 6 respectively.

$$\mu_{i+1} \leq \mu_1 + \mu_2 + \ldots + \mu_j \leq 8\mu_1 + 8\mu_2 + \ldots + 8\mu_j \leq 8^{j-1}\mu_1 + 8^{j-1}\mu_2 + \ldots + 8^{j-1}\mu_j$$

$$\leq \sum_{j=1}^{i} 8^{j-1}\mu_j.$$

Corollary 1.4. *Let* $\mu = (\mu_1^{m_1}\mu_2^{m_2} \ldots \mu_l^{m_l})$ *be a* 9-*tuple complete partition of a positive integer n. Then* $\mu_{i+1} \leq \sum_{j=1}^{i} 9^{j-1}\mu_j$ *where* μ_{i+1} *is the last part of the* 9-*tuple complete partition of an integer.*

Proof. In a 9-tuple complete partition, $n \geq 22$ and $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ and μ_6 should be equivalent to 1, 2, 3, 4, 5 and 6 respectively.

$$\mu_{i+1} \leq \mu_1 + \mu_2 + \ldots + \mu_j \leq 9\mu_1 + 9\mu_2 + \ldots + 9\mu_j \leq 9^{j-1}\mu_1 + 9^{j-1}\mu_2 + \ldots + 9^{j-1}\mu_j$$

$$\leq \sum_{j=1}^{i} 9^{j-1}\mu_j.$$

Corollary 1.5. *Let* $\mu = (\mu_1^{m_1}\mu_2^{m_2} \ldots \mu_l^{m_l})$ *be a* 10-*tuple complete partition of a positive integer n. Then* $\mu_{i+1} \leq \sum_{j=1}^{i} 10^{j-1}\mu_j$ *where* μ_{i+1} *is the last part of the* 10-*tuple complete partition of an integer.*

Proof. In a 10-tuple complete partition, $n \geq 22$ and $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ and μ_6 should be equivalent to 1, 2, 3, 4, 5 and 6 respectively.

$$\mu_{i+1} \leq \mu_1 + \mu_2 + \ldots + \mu_j \leq 10\mu_1 + 10\mu_2 + \ldots + 10\mu_j \leq 10^{j-1}\mu_1 + 10^{j-1}\mu_2 + \ldots + 10^{j-1}\mu_j$$

$$\leq \sum_{j=1}^{i} 10^{j-1}\mu_j.$$

Conclusion

In this paper the complete partitions of the higher order partitioning of septuple, octuple, nonuple and decuple of an integer is identified. From the concept of complete partitions an attempt has been given for the conditions of 7, 8, 9, 10-tuple (septuple, octuple, nonuple, decuple). This work may be extended upto k-tuple complete partitions of integers, to find the generating functions for higher order partitioning.

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